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# On the completeness of photon-added coherent states

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**Abstract**

We demonstrate explicitly the completeness of photon-added coherent states (PACSs), introduced by Agarwal and Tara (Agarwal G S and Tara K 1991 *Phys. Rev.* A **43** 492) and defined, up to normalization, by $(\hat{a}^\dagger)^M|z\rangle$, $M = 0, 1, 2, \ldots$, where $\hat{a}^\dagger$ is the boson creation operator and $|z\rangle$ are conventional Glauber–Klauder coherent states. We find the analytical form of the positive weight function in their resolution of unity by solving the associated Stieltjes power-moment problem. We furnish an example of generation of another set of PACSs which are complete.

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The coherent states, denoted by $|z\rangle$, are special linear combinations of eigenfunctions $|n\rangle$ of a Hermitian operator $\hat{H}$, usually the Hamiltonian, such that all of their coefficients are parametrized with a single complex number $z$.

They have become standard tools in many fields such as condensed matter, quantum optics and field theory [1]. Such states are usually constructed according to a minimal set of fundamental requirements [1, 2]: (i) normalizability, (ii) continuity in label $z$ and (iii) completeness, or the resolution of unity. This last condition follows from the existence of a positive weight function $W(|z|^2)$ such that for $\mathrm{d}^2z \equiv \mathrm{d}(\mathrm{Re}\, z)\,\mathrm{d}(\mathrm{Im}\, z)$ the following equations are valid:

$$\iint_{\mathbb{C}} \mathrm{d}^2z\, |z\rangle\, W(|z|^2)\, \langle z| = I = \sum_n |n\rangle\langle n| \tag{1}$$

where the index $n$ runs either over a whole spectrum of $\hat{H}$ or over some subset of it, depending on the character of state $|z\rangle$. The completeness relations (1) are usually difficult to satisfy. As a consequence the family of truly coherent states is still small in number. Once the weight function $W(|z|^2)$ is known the states $|z\rangle$ can be used, amongst other applications, as a (generally non-orthogonal) basis to obtain the bounds for the finite-temperature partition function through the Lieb–Berezin inequalities [3] ($\beta = \frac{1}{k_\mathrm{B}T}$):

$$\iint_{\mathbb{C}} \mathrm{d}^2z\, W(|z|^2)\, \exp\left(-\beta\, \langle z|\hat{H}|z\rangle\right) \leqslant \mathrm{Tr}\, \mathrm{e}^{-\beta\,\hat{H}}. \tag{2}$$

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Recently progress has been achieved by finding the weight functions for some specific choices of the states $|z\rangle$ (see [4–6]).

The purpose of this paper is to demonstrate the completeness of an important special family of states constructed by adding photons to a conventional Glauber–Klauder coherent state $|z\rangle$ [1]. This last state is defined for the Hamiltonian of the linear harmonic oscillator $\hat{H}_0 = \hat{a}^\dagger\hat{a}$ (with $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{H}_0|n\rangle = n|n\rangle$, $\langle n|n'\rangle = \delta_{n,n'}$, $n = 0, 1, \ldots, \infty$) as

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \exp(z\hat{a}^\dagger)|0\rangle \tag{3}$$

$$= \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \tag{4}$$

where in equation (3), $|0\rangle$ is the ground state of $\hat{H}_0$ (the vacuum). The photon-added states were introduced by Agarwal and Tara [7], who also have exhaustively examined their non-classical fluctuation properties. The experimental realizations and applications of these states were objects of extensive studies recently [8, 9].

The normalized photon-added coherent state (PACS) reads [7]

$$|z; M\rangle = \mathcal{N}_M^{-\frac{1}{2}}(|z|^2)\,(\hat{a}^\dagger)^M|z\rangle \tag{5}$$

$$= [M!\,L_M(-|z|^2)]^{-\frac{1}{2}}(\hat{a}^\dagger)^M|z\rangle \longrightarrow \begin{cases} |z\rangle & \text{if} \quad M \longrightarrow 0, \quad z = \text{const} \\ |M\rangle & \text{if} \quad M = \text{const}, \quad z \longrightarrow 0 \end{cases} \tag{6}$$

where $L_M(y)$ is the $M$th Laguerre polynomial. Observe that $L_M(-y)$ is always positive. Equation (6) means that $|z; M\rangle$ interpolates between the coherent state $|z\rangle$ and the Fock or number state $|M\rangle$. The development of the PACS in terms of number states $|n\rangle$ follows from equations (4) and (6):

$$|z; M\rangle = [\mathrm{e}^{|z|^2} M!\,L_M(-|z|^2)]^{-\frac{1}{2}}\,(\hat{a}^\dagger)^M \sum_{n=0}^{\infty} \frac{(\hat{a}^\dagger)^n z^n}{n!}|0\rangle \tag{7}$$

$$= [\mathrm{e}^{|z|^2} M!\,L_M(-|z|^2)]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\sqrt{(n+M)!}}{n!} z^n |n+M\rangle. \tag{8}$$

For fixed $z$ the state $|z; M\rangle$ is normalized for any finite $M$. We note that $|z; M\rangle$ is a linear combination of all the number states starting with $n = M$; i.e. the first $M$ number states, $n = 0, 1, \ldots, M-1$, are absent from the wavefunction $|z; M\rangle$. The unity operator in this space is to be written as

$$I_M = \sum_{n=M}^{\infty} |n\rangle\langle n| = \sum_{n=0}^{\infty} |n+M\rangle\langle n+M| \tag{9}$$

and the resolution of unity reads

$$\iint_{\mathbb{C}} \mathrm{d}^2 z\,|z; M\rangle\, W_M(|z|^2)\,\langle z; M| = I_M = \sum_{n=0}^{\infty} |n+M\rangle\langle n+M|. \tag{10}$$

To obtain the conditions for the sought-for positive function $W_M(|z|^2)$ we substitute equation (8) into (10), use $z = |z|\mathrm{e}^{\mathrm{i}\theta}$ and perform the $\theta$ integration, thus projecting out the off-diagonal terms $|n\rangle\langle n'|$. One is then left with the conditions, for $x \equiv |z|^2$,

$$\pi \frac{(n+M)!}{(n!)^2} \int_0^\infty x^n \left[\frac{W_M(x)}{\mathrm{e}^x\, M!\, L_M(-x)}\right] \mathrm{d}x = 1 \qquad n = 0, 1, \ldots, \infty. \tag{11}$$

Introducing the function $\tilde{W}_M(x) \equiv \pi \frac{W_M(x)}{e^x L_M(-x)}$ equation (11) boils down to the infinite set of equations for $\tilde{W}_M(x)$:

$$\int_0^\infty x^n \tilde{W}_M(x)\, dx = M!\, \frac{(n!)^2}{(n+M)!} \equiv \rho_M(n) \qquad n = 0, 1, \ldots, \infty \tag{12}$$

which is the classical Stieltjes power-moment problem [10] for $\tilde{W}_M(x)$, with the set of moments $\rho_M(n)$. The task of proving the existence of a positive solution $\tilde{W}_M(x)$ is prohibitively difficult as it amounts to proving the positivity of two special series of all upper-left-corner Hankel–Hadamard determinants formed out of $\rho_M(n)$ [10].

We shall overcome this difficulty by constructing by auxiliary means the solution of equations (12) and observing that it is positive. The actual approach to solve equations (12) for $\tilde{W}_M(x)$ is to extend the natural values of $n$ to complex $s$ such that $n \longrightarrow s - 1$, and then to observe that equations (12) can be interpreted as the Mellin transform [11], defined as $\mathcal{M}[f(x); s] = \int_0^\infty x^{s-1} f(x)\, dx \equiv f^*(s)$, and consequently the inverse Mellin transform is $\mathcal{M}^{-1}[f^*(s); x] = f(x)$.

This identification has been recently applied to find solutions of a number of moment problems arising in connection with different kinds of generalized coherent state [4–6, 12].

The above reparametrization allows one to rewrite equation (12) as

$$\mathcal{M}[\tilde{W}_M(x); s] = \int_0^\infty x^{s-1} \tilde{W}_M(x)\, dx = M!\, \frac{\Gamma(s)^2}{\Gamma(s+M)} \tag{13}$$

$$\equiv M!\, \Gamma\left[\begin{matrix} s, s \\ s+M \end{matrix}\right] \qquad \operatorname{Re} s > 0. \tag{14}$$

From equation (14) the solution of equation (12) is formally given by

$$\tilde{W}_M(x) = M!\, \mathcal{M}^{-1}\left[\Gamma\left[\begin{matrix} s, s \\ s+M \end{matrix}\right]; x\right]. \tag{15}$$

This last inverse Mellin transform is known, compare the formula 4(1) on p 285 of [13] or the formula 8.4.46.7 on p 716 of [14]; the weight function $\tilde{W}_M(x)$ is equal to

$$\tilde{W}_M(x) = M!\, e^{-x}\, \Psi(M, 1; x) \tag{16}$$

where $\Psi(a, c; x)$ is Tricomi's confluent hypergeometric function [14]. As $\Psi(0, 1; x) = 1$, the case $M = 0$ reproduces the known weight function $\sim e^{-x}$ of the conventional coherent state $|z\rangle$ [1]. $\tilde{W}_M(x)$ can be expressed by known functions in several ways. We shall use here Tricomi's integral [13],

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-xt}\, t^{a-1}}{(1+t)^{1+a-c}}\, dt \qquad \operatorname{Re} a, \operatorname{Re} x > 0 \tag{17}$$

which with equation (16) yields

$$\tilde{W}_M(x) = M\, e^{-x} \int_0^\infty \frac{e^{-xt}\, t^{M-1}}{(1+t)^M}\, dt \qquad M \geqslant 1, \quad x \geqslant 0 \tag{18}$$

which is a positive function. We stress that, due to the restriction on $a$ in equation (17), the expression (18) for $\tilde{W}_M(x)$ applies only for $M \geqslant 1$.

In equation (18) we introduce a new variable $y = 1 + t$, expand the resulting binomial and integrate term by term. We obtain finally

$$\tilde{W}_M(x) = M \sum_{p=0}^{M-1} (-1)^p \binom{M-1}{p} \mathrm{E}_{p+1}(x) \qquad M \geqslant 1, \quad x > 0 \tag{19}$$

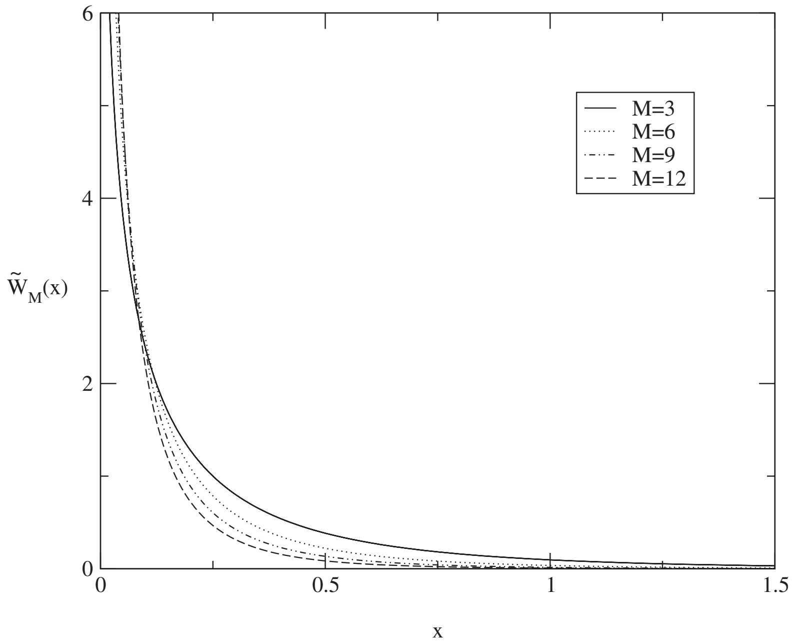


**Figure 1.** The weight function $\tilde{W}_M(x)$ of equation (19) versus $x$ for different values of $M$.

and

$$W_M(x) = \frac{M}{\pi}\,\mathrm{e}^x\,L_M(-x)\sum_{p=0}^{M-1}(-1)^p\binom{M-1}{p}\mathrm{E}_{p+1}(x) \qquad M \geqslant 1, \quad x > 0. \tag{20}$$

In equations (19) and (20), $\mathrm{E}_k(z)$ is a generalized exponential integral of order $k$ [15] defined as $\mathrm{E}_k(z) = \int_1^\infty \mathrm{e}^{-zt}/t^k\,\mathrm{d}t$ $(z > 0)$.

In figures 1 and 2 we have displayed the functions $\tilde{W}_M(x)$ and $W_M(x)$ respectively, for different values of $M$. It is seen that $\tilde{W}_M(x)$ has an (integrable) singularity at $x = 0$, whereas $W_M(x)$ is not integrable as for $M = \text{const}$, $\lim_{x\longrightarrow\infty} W_M(x) = 1/\pi$, which is equal to $W(x)$ for conventional coherent states [1]. This last result signifies that in $|z; M\rangle$ of equation (8) in the limit $|z|^2 \longrightarrow \infty$ the missing states with $n = 0, 1, \ldots, M-1$ are of no importance as then only the states with $n \gg M$ have a dominant contribution.

The form of the $\rho_M(n)$ of equation (12) permits one to verify the unicity of the solution $\tilde{W}_M(x)$. To this end we apply the Carleman criterion [10]: if $S = \sum_{n=0}^{\infty}[\rho_M(n)]^{-\frac{1}{2n}}$ diverges, then the solution of the moment problem is unique. The substitution of $\rho_M(n)$ to $S$ shows that indeed $S = \infty$, thus confirming that $\tilde{W}_M(x)$ is a unique solution for the set of moments $\rho_M(n)$ of equation (12). We conclude that the PACSs are a well defined family of coherent states as they fulfil the minimal set of requirements alluded to above.

The general structure of the expansion (7) and of the associated moment problem (12) suggest a prescription to generate the PACSs which are different from $|z; M\rangle$ of equation (8). We sketch one such case here. We are asking the following question: can one choose a quantum state $|z\rangle' \neq |z\rangle$ such that by adding $M$ photons to it we end up with $|z; M\rangle' \neq |z; M\rangle$ such that $|z; M\rangle'$ are a complete set, in a subspace where all the states $|0\rangle, \ldots, |M-1\rangle$ are absent? Clearly this means that $|z\rangle'$ is so tailored that the moment problem generated by $|z; M\rangle'$ has a positive weight function. Note that the set of $|z\rangle'$ may or may not be complete. A case in point

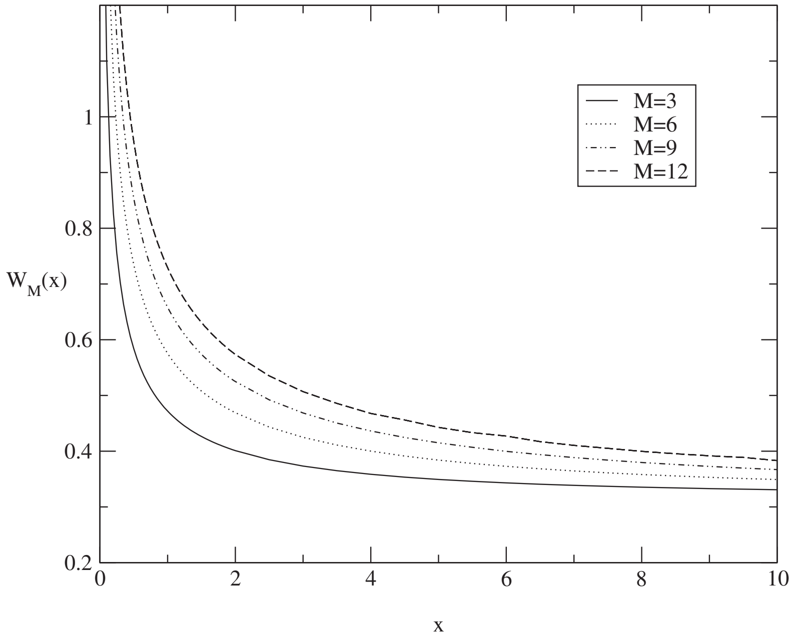


**Figure 2.** The weight function $W_M(x)$ of equation (20) versus $x$ for different values of $M$.

is an unnormalized (but normalizable) state $|z\rangle'$ defined by

$$|z\rangle' = \sum_{n=0}^{\infty} \frac{z^n}{n!}|n\rangle. \tag{21}$$

This choice of $|z\rangle'$ is perhaps the simplest extension of conventional coherent states of equation (4), retaining their holomorphic character. Many other extensions are possible such as $\sum_{n=0}^{\infty} \frac{z^n}{(n!)^2}|n\rangle$, $\sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\frac{5}{2}}}|n\rangle$ etc. Every state of this type will lead to a certain moment problem (compare equation (29) below) which should be investigated separately.

We construct now a modified normalized PACS, using equation (21), as

$$|z; M\rangle' = [M!\mathcal{N}'_M(|z|^2)]^{-\frac{1}{2}} (\hat{a}^\dagger)^M |z\rangle' \tag{22}$$

$$= [\mathcal{N}'_M(|z|^2)]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sqrt{\frac{(n+M)!}{M!}} \frac{z^n}{(n!)^{\frac{3}{2}}} |n+M\rangle \tag{23}$$

whose normalization is ($x \equiv |z|^2$)

$$M!\mathcal{N}'_M(x) = \sum_{n=0}^{\infty} \frac{(n+M)!}{(n!)^3} x^n = {}_1F_2(1+M; 1, 1; x) \tag{24}$$

with particular cases

$$\mathcal{N}'_1(x) = I_0(2\sqrt{x}) + \sqrt{x}\, I_1(2\sqrt{x}) \tag{25}$$

$$\mathcal{N}'_2(x) = 2\left(\left(1+\frac{x}{2}\right) I_0(2\sqrt{x}) + \frac{3}{2} I_1(2\sqrt{x})\right) \tag{26}$$

$$\mathcal{N}'_3(x) = \tfrac{1}{6}((6+7x)\, I_0(2\sqrt{x}) + \sqrt{x}\,(11+x)\, I_1(2\sqrt{x})) \tag{27}$$

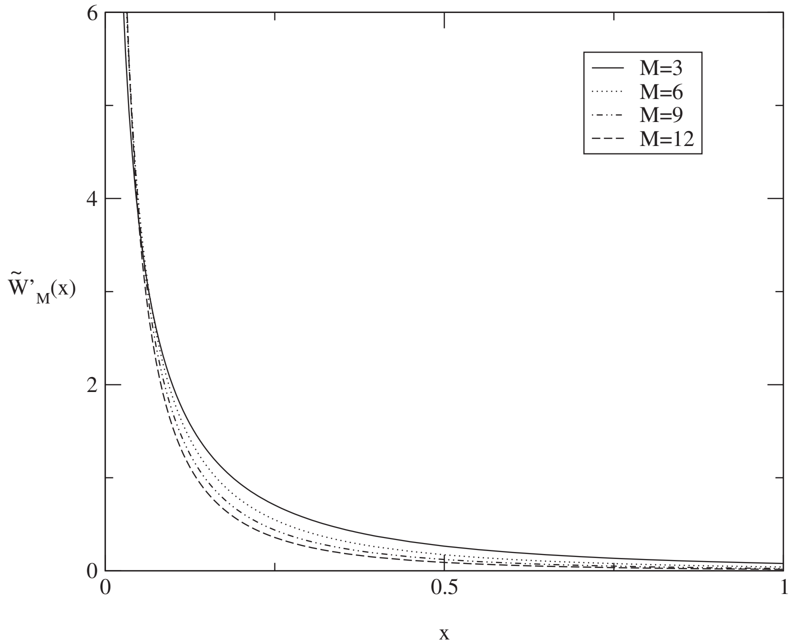


**Figure 3.** The weight function $\tilde{W}'_M(x)$ of equation (34) versus $x$ for different values of $M$.

etc, where ${}_1F_2(a; b, c; x)$ is the hypergeometric function and $I_0$ and $I_1$ are modified Bessel functions of first kind. We plug now the normalized states $|z; M\rangle'$ into their resolution of unity condition:

$$\int\!\!\int_{\mathbb{C}} \mathrm{d}^2 z\, |z; M\rangle' \, W'_M(|z|^2)\, '\langle z; M| = I_M \tag{28}$$

and obtain, for $\tilde{W}'_M(x) = \pi \frac{W'_M(x)}{\mathcal{N}'_M(x)}$, the following Stieltjes power-moment problem:

$$\int_0^\infty x^n \, \tilde{W}'_M(x)\, \mathrm{d}x = M!\, \frac{(n!)^3}{(n+M)!} \qquad n = 0, 1, \ldots, \infty. \tag{29}$$

The function $\tilde{W}'_M(x)$ is positive. This can be seen by rewriting (29) as

$$\mathcal{M}[\tilde{W}'_M(x); s] = M!\, \frac{\Gamma(s)^2}{\Gamma(s+M)}\, \Gamma(s) \tag{30}$$

which, through the use of the convolution property of the inverse Mellin transform [11]

$$\mathcal{M}^{-1}[f^*(s) \cdot g^*(s); x] = \int_0^\infty \frac{1}{t}\, f\left(\frac{x}{t}\right) g(t)\, \mathrm{d}t \tag{31}$$

gives

$$\tilde{W}'_M(x) = M! \int_0^\infty \frac{\mathrm{e}^{-(t+\frac{x}{t})}}{t}\, \Psi(M, 1; t)\, \mathrm{d}t \tag{32}$$

which is an integral of a positive function. A way to obtain a more explicit form of $\tilde{W}'_M(x)$ is to relate it to Meijer's $G$-function $G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$ through [13]

$$\mathcal{M}\left[M!\, G^{3,0}_{1,3}\left(x \,\middle|\, \begin{matrix} M \\ 0, 0, 0 \end{matrix}\right); s\right] = M!\, \frac{\Gamma(s)^3}{\Gamma(s+M)} \equiv M!\, \Gamma\left[\begin{matrix} s, s, s \\ s+M \end{matrix}\right]. \tag{33}$$

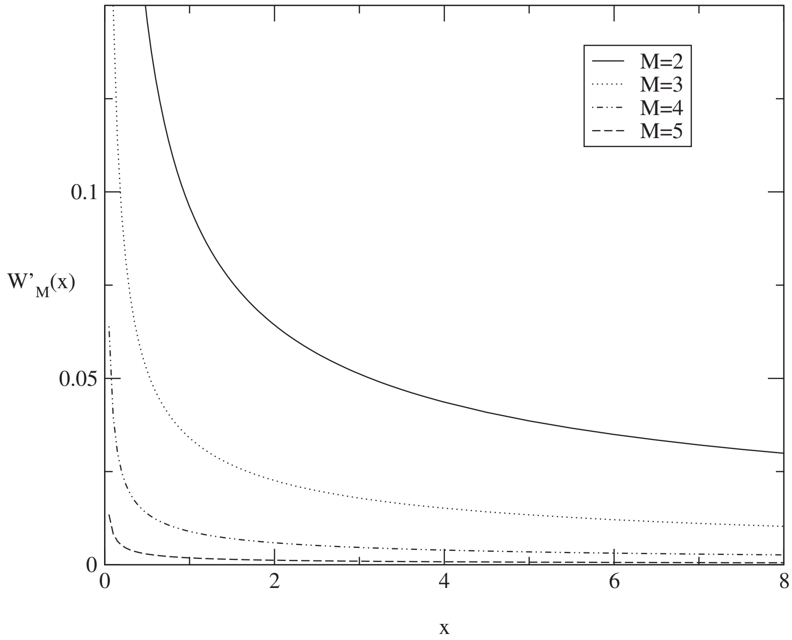


**Figure 4.** The weight function $W'_M(x)$ of equation (28) versus $x$ for different values of $M$.

Apparently $G^{3,0}_{1,3}\left(x\,\middle|\,\begin{matrix} M \\ 0,0,0\end{matrix}\right)$ cannot be simply expressed by known functions but its series representation is known, yielding

$$\begin{aligned}\tilde{W}'_M(x) = \frac{M!}{2}\sum_{n=0}^{\infty}&(-1)^n[(\ln(x))^2+\pi^2+2\ln(x)(\psi(M-n)-3\,\psi(n+1))\\ &-3\,\psi^{(1)}(n+1)+\psi(M-n)^2-6\,\psi(M-n)\,\psi(n+1)\\ &+9\,\psi(n+1)^2-\psi^{(1)}(M-n)]\frac{x^n}{\Gamma(M-n)\Gamma(n+1)^3}\end{aligned} \tag{34}$$

where $\psi(x)$ is the digamma function and $\psi^{(1)}(x)$ is the polygamma function of order one (compare [15]). In equation (34) the factor $\sim[\Gamma(M-n)]^{-1}$ automatically truncates the series multiplying $[\ln(x)]^2+\pi^2$ and simplifies the $\psi$-terms. $\tilde{W}'_M(x)$ display a singularity at $x=0$ as the functions $\tilde{W}_M(x)$ of equation (16) do. The Carleman criterion applied to equation (29) again indicates the unique character of the solution of equation (34). The weight functions $\tilde{W}'_M(x)$ and $W'_M(x)$ are represented in figures 3 and 4 respectively, for different values of $M$. We conclude that the modified PACSs of equation (23) form a complete set.

A legitimate question is how to construct photon-substracted complete coherent states by acting with the boson annihilation operator on some state $|\tilde{z}\rangle$, thus producing the normalized state $|\tilde{z};M\rangle = \tilde{\mathcal{N}}_M^{-\frac{1}{2}}(|z|^2)\,(\hat{a})^M|\tilde{z}\rangle$. It is clear that the state $|\tilde{z}\rangle$ must be different from $|z\rangle$ of equation (4) as this last state is an eigenstate of $(\hat{a})^M$, $(\hat{a})^M|z\rangle = z^M\,|z\rangle$ and thus the resulting $|\tilde{z};M\rangle$ would not be normalizable at $z=0$. In fact the requirement that $|\tilde{z};M\rangle$ constitute a complete set imposes quite stringent constraints on the possible form of $|\tilde{z}\rangle$ and this will be a subject of a separate study.

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